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Basin bifurcation in quasiperiodically forced systems

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In this paper we study quasiperiodically forced systems exhibiting fractal and Wada basin boundaries. Specifically, by utilizing a class of representative systems, we analyze the dynamical origin of such basin boundaries and we characterize them. Furthermore, we find that basin boundaries in a quasiperiodically driven system can undergo a unique type of bifurcation in which isolated ''islands'' of basins of attraction are created as a system parameter changes. The mechanism for this type of basin boundary bifurcation is elucidated. [S1063-651X(98)07909-4]

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I. INTRODUCTION

Many physical, chemical, biological, and engineering processes are known to possess multiple coexisting final states. Often, these processes can be modeled by either N-dimensional continuous flows $d\mathbf{x}/dt=\mathbf{F}(\mathbf{x},p)$ or N-dimensional discrete maps $\mathbf{x}_{n+1}=\mathbf{F}(\mathbf{x}_n,p)$, where $\mathbf{x}\in\mathbf{R}^N$ is the state variable, $\mathbf{F}$ is a nonlinear function that has N components, and p is a system parameter. Multiple coexisting final states mean that for a given parameter value p, different choices of the initial condition $\mathbf{x}_0$ can lead to distinctly different asymptotic attractors, each with its own basin of attraction. The basin of attraction of an attractor is the set of initial conditions in the phase space that asymptote to the attractor. The boundaries that separate different basins of attraction are the *basin boundaries*, which can be either smooth or fractal [1–7]. When the boundary is smooth, its box dimension D is one less than that of the phase space, i.e., $D=N-1$. For a fractal basin boundary, its dimension D is a fractional number that satisfies $(N-1)<D<N$, where the number $\alpha=N-D<1$ is the so-called uncertainty exponent [8]. More recently, common fractal basin boundaries of more than two basins of attractions, the *Wada basin boundaries*, have been identified in dynamics and studied [9–11]. It has been known that fractal and Wada basin boundaries lead to a *final state sensitivity* [1–7,9–11]. That is, for a specific parameter setting and initial condition, no reliable computation can be made to predict the system's asymptotic attractor.

So far, to our knowledge, the study of fractal and Wada basin boundaries has been restricted to dynamical systems with no external driving or periodically driven systems [1–7,9–11]. A quite important class of dynamical systems are the quasiperiodically forced systems, systems driven at two or more incommensurate frequencies. These systems are of interest because they can occur in physical situations such as the dynamics of a quasiperiodically driven superconducting quantum interference device [12], or in situations where some chemical or biological oscillators are driven by two or more periodic signals whose frequencies are incommensurate [13]. In engineering, quasiperiodically driven systems are also of interest [14].

The purpose of this paper is to address the question of how fractal and Wada basins evolve in quasiperiodically forced systems as the system's parameters are varied. Our approach will be to study a class of representative systems: the quasiperiodically forced maps [15]. We choose to study maps because they exhibit many fundamental phenomena of the quasiperiodically driven flows such as strange nonchaotic attractors [16,17,15,18,12], yet the analyses and computations involved are greatly simplified. We find that multiple coexisting attractors, fractal and Wada basin boundaries are common in the sense that they occur in wide parameter regions of the systems studied. In particular, we study a basin boundary bifurcation that characterizes a sudden change of the basin boundary as a parameter changes. We find that in quasiperiodically forced systems, Wada basin boundaries can undergo a unique type of bifurcation in which isolated and islandlike basins are created in the originally open basins. We give a detailed analysis to account for this type of bifurcation.

The rest of the paper is organized as follows. In Sec. II we describe our model, show numerical evidence for the presence of fractal and Wada basin boundaries, and quantify these boundaries by using the uncertainty exponent [1,2]. In Sec. III we present an analysis for the occurrence of Wada basins. In Sec. IV we describe and analyze the phenomenon of a basin boundary bifurcation in quasiperiodically driven systems. Discussions are presented in Sec. V.

II. NUMERICAL EVIDENCE

Our model system is the following class of two-dimensional maps:

$$\begin{aligned}\theta_{n+1}&=\theta_n+\omega \mod 1,\\ x_{n+1}&=M(x_n)+F(\theta_n),\end{aligned}\tag{1}$$

where $M(x)$ is a nonlinear map that can exhibit chaos, and $F(\theta_n)$ models an external driving. We consider the simplest type of driving: $F(\theta_n)=a\cos(2\pi\theta_n)$, where a is the driving amplitude. The driving is quasiperiodic when the parameter ω in the first line of Eq. (1) is chosen to be an irrational number. We choose ω to be the inverse of the golden mean:

$\omega=(\sqrt{5}-1)/2$ throughout this paper. In order to study basin boundaries, it is necessary to choose the map $M(x)$ so that Eq. (1) possesses multiple coexisting attractors. For illustrative purpose, we choose $M(x)$ to be the three-times iterated version of the logistic map, that is, $M(x)=f^{(3)}(x)$ where $f(x)=rx(1-x)$. For simplicity we choose the parameter r in the logistic map $f(x)$ so that it is in a period-3 window and, hence, the map $M(x)$ possesses three isolated simple attractors, each with its own basin of attraction. We choose the parameter r so that these attractors are fixed-point attractors. When the quasiperiodic forcing is present $(a\neq 0)$, the map Eq. (1) possesses then three isolated attractors in the two-dimensional phase space (θ,x).

We now present numerical evidence for the existence of fractal and Wada basin boundaries in Eq. (1). The key observation is that for the one-dimensional map $M(x)$, the boundaries between the basins of attraction of the three attractors are Cantor sets (fractal) [2,4,19]. Under the quasiperiodic forcing at small amplitudes, the three fixed-point attractors in the one-dimensional phase space of $M(x)$ are transformed into three attractors (either quasiperiodic, strange nonchaotic, or chaotic) [15]. The boundaries between the basins of attraction of these attractors we expect to be topologically the fractal boundary sets that already exist in the map $M(x)$ cross with a circle (in the θ direction). Therefore we expect the basin boundaries between the three quasiperiodic attractors in Eq. (1) to be fractal too. Figures 1(a)–1(c) show for $r=3.833$ and $a=0.0015$, the basins of attraction at three different scales, where Figs. 1(b) and 1(c) are successive enlargements of Fig. 1(a). In Fig. 1(a), only one of the three attractors is shown, the one whose basin is denoted by white dots in the figure. The basins of the other two attractors are indicated by black and gray dots, respectively. Figures 1(a)–1(c) suggest the existence of fractal basin boundaries in Eq. (1).

We now characterize, quantitatively, the fractal Wada basin boundary in Figs. 1(a)–1(c). It has been known that fractal basin boundaries pose a fundamental difficulty in the prediction of the asymptotic attractor of the system [1,2] because of the interwoven fractal structure of the basins of attraction and because of the inevitable error in the specification of initial conditions and system parameters. This is called the *final state sensitivity* [1,2]. Let ϵ be such an error. Then the probability for two initial conditions, of ϵ distance apart, to asymptote to different attractors scales with ϵ as

$$P(\epsilon)\sim\epsilon^{\alpha}, \tag{2}$$

where the scaling exponent is the uncertainty exponent α, with $0<\alpha\leqslant 1$ [1,2]. Since $P(\epsilon)$ can be regarded as the error to predict the asymptotic attractor with finite measurement precision ϵ, we see that a significant improvement in the precision, or a substantial reduction in ϵ, usually yields only a modest decrease in the prediction error $P(\epsilon)$ if α is less than one. In the extreme case where $\alpha\approx 0$, many orders of magnitude of reduction in ϵ would yield essentially no reduction in $P(\epsilon)$, a situation which is common in high-dimensional dynamical systems [20,21] or in systems with riddled basins [22].

To compute the uncertainty exponent associated with the boundary between two of the basins, we choose a number of

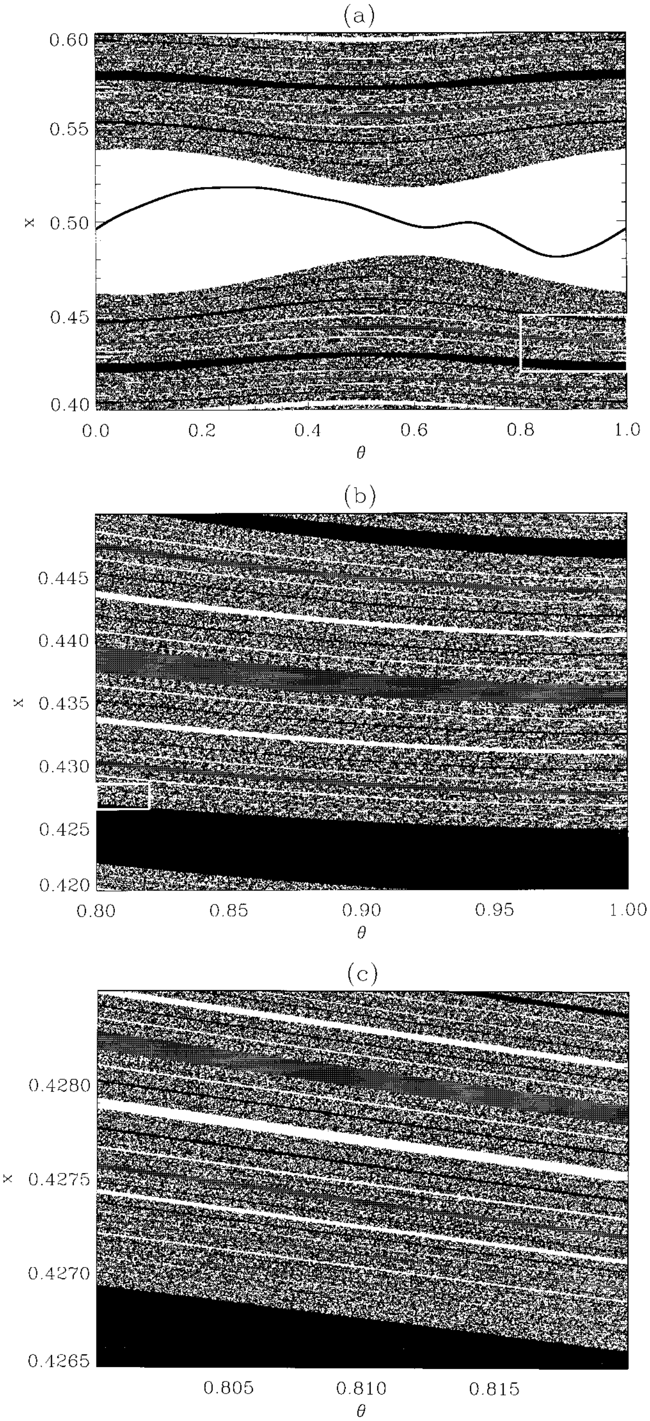

FIG. 1. (a) The basins of attraction of the three attractors for the quasiperiodically forced logistic map. The figures show only the ''middle'' part of the phase space containing one of the attractors. The white region belongs to the basin of the attractor in the figure, while the black and gray regions belong to the basins of the other two attractors. The parameter setting is $r=3.833$, $a=0.0015$. The basin boundaries are apparently fractal. (b) and (c) Two successive enlargements of the rectangles indicated in white.

values of ϵ in the range $10^{-14}\leqslant\epsilon\leqslant 10^{-3}$. For each value of ϵ, we choose random initial conditions in the region $0\leqslant\theta\leqslant 1$ and $0\leqslant x\leqslant 1$ in one of the two basins. We perturb each one by ϵ, and we then determine if the perturbed initial condition asymptotes to the same attractor as the unperturbed one. If yes, the pair is called certain with respect to small

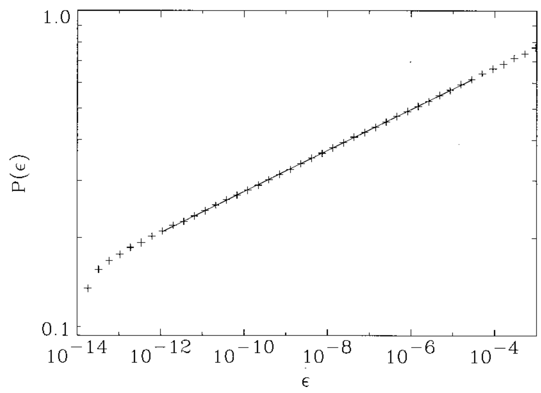

FIG. 2. $\ln P(\epsilon)$ versus $\ln \epsilon$ for the basin boundary between the white and black basins in Fig. 1. A linear fit yields the following uncertainty exponent: $\alpha \approx 0.07$, indicating that the dimension of the particular basin boundary is $D = 2 - \alpha \approx 1.93$.

perturbation ϵ. Otherwise it is uncertain. The uncertain probability $P(\epsilon)$ is approximately the fraction of, say, 1000 uncertain initial condition pairs among total pairs chosen. Figure 2 shows, on a log-log scale, $P(\epsilon)$ versus ϵ. We see that the plot can be well fitted by a straight line, indicating the scaling relation (2). We obtain $\alpha \approx 0.07$. The fractal dimension of the basin boundary between the two basins is then $D = 2 - \alpha \approx 1.93$, which is close to the phase-space dimension. This indicates that the basin boundary separating the two attractors has an arbitrarily fine-scale structure and, for all practical purposes, it is very difficult to predict the asymptotic attractor for a given initial condition. We get the same dimension if one looks at the boundary between any other two basins.

For the parameter setting in Figs. 1(a)–1(c), there are apparently three attractors with fractal basin boundaries separating their basins. An interesting question is then whether there exists a *common* fractal boundary among the three basins of attraction. Such a common boundary is said to possess the Wada property and is hence called a *Wada basin boundary* [9]. In the following we shall argue that there is indeed a Wada boundary in Figs. 1(a)–1(c). In order to test whether the fractal basin boundary is common to the three basins, we calculate the dimension of the union of the pairwise boundaries. The numerical computation shows that this dimension is the same as the dimensions for the pairwise boundaries.

III. WADA BASIN BOUNDARY

We first review the concept of a Wada basin boundary. Consider three basins of attraction B_1, B_2, and B_3. The basin B_1 is a Wada basin if every point in the boundary of B_1 is also in the boundary of B_2 and B_3. The same definition holds for B_2 and B_3. To have the Wada property, the three basins must be pairwise disjoint. Such a geometric construction of three regions in which every boundary point is a boundary point of all three regions was first conceived by the Dutch mathematician Brouwer in 1910 and independently by the Japanese mathematician Yoneyama in 1917, which was called the ''Lakes of Wada'' in Ref. [23]. The natural occurrence of the Lakes of Wada phenomenon in chaotic dynamical systems (Wada basins) was first identified and analyzed by Kennedy and Yorke [9]. They found that the exotic Wada-basin phenomenon occurs quite commonly even in low-dimensional dynamical systems such as three-dimensional flows and two-dimensional maps. Recently Nusse and Yorke showed rigorously that Wada basins can occur generally in dynamical systems [10].

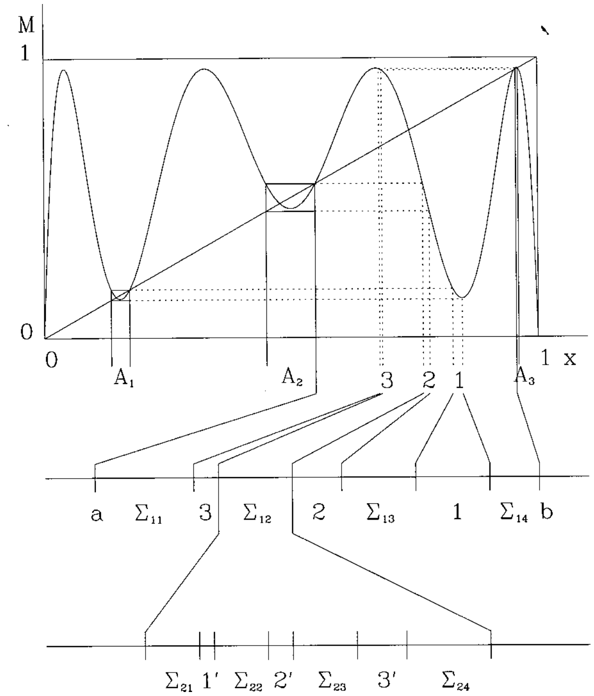

FIG. 3. Plot of the map $M(x)$. There are three square regions that correspond to three one-dimensional subintervals in [0,1] in which the three fixed-point attractors lie. Analysis of the preimages of these subintervals leads to the conclusion that the basin boundary in Fig. 1 has the Wada property.

To argue for the existence of a fractal Wada-basin boundary, we consider the case where $a \gtrsim 0$ so that the basin boundary is a topological transformation of the Cantor boundary set $\mathbf{C}$ in the one-dimensional map $M(x)$ to $\mathbf{C} \times \mathbf{S}^1$, as shown in Figs. 1(a)–1(c), where $\mathbf{S}^1$ is the circle in the θ direction. It thus suffices to argue that the Cantor boundary set in $M(x)$ possesses the Wada property. Referring to Fig. 3, a plot of the map $M(x)$, we see there are three square regions that correspond to three one-dimensional subintervals in [0,1], each one with a fixed-point attractor. Denote these subintervals by A_1, A_2, and A_3. The boundary between the three basins of attraction must then lie in the one-dimensional set which is the complement set in [0,1] of the subintervals A_1, A_2, and A_3. Concentrating on one of the complement intervals, say $[a,b]$, we see that there are three subintervals in $[a,b]$, denoted by 1, 2, and 3, which are the preimages of A_1, A_2, and A_3, respectively. The interval $[a,b]$ thus contains all three basins and contains the complement set of the joint set of subintervals 1, 2, and 3 in $[a,b]$. This complement set consists of four subintervals, denoted by Σ_{11}, Σ_{12}, Σ_{13}, and Σ_{14}, respectively, as shown in Fig. 3. Now look at one of these four subintervals, say Σ_{12}, the one in between 3 and 2. We see that there are three subintervals in Σ_{12}, denoted by 1′, 2′, and 3′, respectively, which map to A_1, A_2, and A_3 in two iterations. The subinterval Σ_{12},

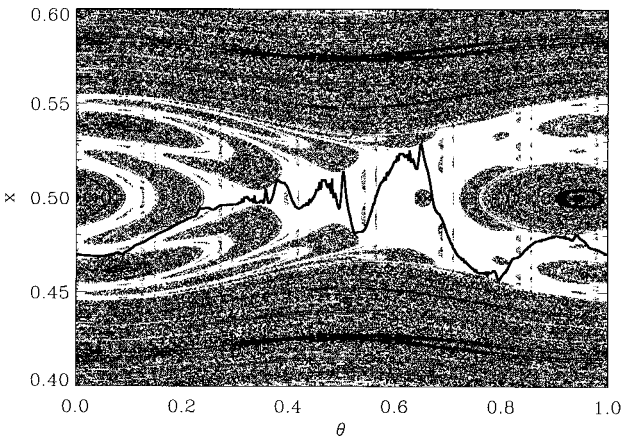

FIG. 4. Basin structures after a basin boundary bifurcation. The parameter setting is $r=3.846$ and $a=0.0024$. We see that there are ''islands'' of new basins of other attractors in the originally open basin of one attractor.

which is smaller than the original interval $[a,b]$, contains all three basins. In a similar fashion, it is easy to see that there are four still smaller subintervals Σ_{21}, Σ_{22}, Σ_{23}, and Σ_{24} in Σ_{12} that contain the basin boundary (see Fig. 3). Any of these smaller subintervals must contain all three basins. By examining the nth preimages of the subintervals A_1, A_2, and A_3 in the limit $n\to\infty$, we see that an arbitrarily small subinterval Σ_{nj} $(j=1,2,3,4)$ must contain all three basins. The boundary between the three basins must then be unique, fractal, and Wada. The same must also be true for the basins shown in Figs. 1(a)–1(c) since the basin boundary is simply a Cantor set of circles ($\mathbf{C}\times\mathbf{S}^1$). Furthermore, since such a Cantor set has a unique dimension [24], the dimension of the basin boundary is also unique.

IV. BASIN BOUNDARY BIFURCATION

Besides the basin boundary consisting of a Cantor set of invariant circles ($\mathbf{C}\times\mathbf{S}^1$), another type of basin boundary can occur in quasiperiodically forced systems. In this case, the basins of attraction of one attractor have isolated ''islands'' immersed in the basins of the other attractors. Figure 4 shows an example for such a basin. The formation of those islands is a result of a sudden change in the structure of the basins of attraction as a system's parameter is changed. This change can be considered as a basin boundary bifurcation occurring at special values of the parameters.

We address the following questions: How can basin boundary bifurcations occur in quasiperiodically driven systems and what are the unique characteristics of such bifurcations? To gain insight, we refer to Fig. 3, the plot of the one-dimensional map $M(x)$ under no driving. In the figure, there are three square regions in which the three attractors lie. The one-dimensional subintervals A_1, A_2, and A_3 belong entirely to the basins of the three attractors. This is due to the fact that the critical points of the map in the three squares are completely in the squares. Now imagine we turn on the quasiperiodic forcing. At small forcing amplitude, the critical points are still in the square so that the subintervals A_1, A_2, and A_3 are still open basins of the three attractors. At different locations of θ, the driving is different. Thus the lengths of the subintervals A_1, A_2, and A_3 are different for different θ values but, nonetheless, the lengths change smoothly due to the smooth driving function $a\,\cos 2\pi\theta$ used, as shown in Fig. 1(a) by the large white region about $x=0.5$. As the forcing amplitude increases, at some locations of θ the driving term $a\,\cos 2\pi\theta$ is larger so that at these locations, the critical points of the map $M(x)$ are no longer contained in the squares. When this happens, a subinterval, say A_2, contains part of the basins of the attractors that are in A_1 and A_3. In this sense, the basin of the attractor in A_2, which is originally connected, now invades the basins of the other attractors. In the two-dimensional phase space (θ,x), we then expect to see complicated basin structures in the originally open basins. In particular, since the effect of forcing is different at different θ values, the newly created basins in the originally open basins form an ''island'' structure, as shown in Fig. 4 for $r=3.846$ and $a=0.0024$. The uncertainty exponent for the basin structure in Fig. 4 is estimated to be $\alpha\approx 0.05$, indicating that the dimension of the fractal Wada boundary is approximately 1.95, which is also close to the phase-space dimension. This islandlike basin structure created after a basin boundary bifurcation is a unique feature of noninvertible systems.

To understand further why basin boundary bifurcations occur for Eq. (1), we employ the concept of critical curves. Critical points and critical curves play an important role in the localization of singularities of the invariant measure of chaotic attractors [25]. Certain phase transitions in nonlinear systems such as band merging or interior crisis can also be understood by examining the dynamics of the critical points in the map [26]. The concept of critical curves has also been used to argue the loss of connectedness of the basins of attractions in two-dimensional noninvertible dissipative maps [27]. The critical points of a map are the iterations of the local extrema. In the logistic map, the critical points are the images of $x=0.5$. In the period-3 window, the critical points are located in the three square regions, as shown in Fig. 3. Because of the θ dynamics, one has critical curves. Consider the critical curve defined by $x=0.5$, $0\leqslant\theta<1$. A special property of this curve is that the determinant of the Jacobian matrix of Eq. (1) evaluated along it is equal to zero. When the quasiperiodic driving is zero, there are critical curves which are straight lines in the θ direction, as shown in Fig. 5(a). They are located in the three basins. When the quasiperiodic forcing is increased from zero, they become wavylike shaped, as shown in Figs. 5(b) and 5(c). The curves become more convoluted as the number of iterations increases. Before the basin boundary bifurcation, the higher iterates of the curves do not touch the basin boundaries and they remain in the vicinities of the three attractors. At some critical forcing amplitude, the higher iterates of the critical curves and the basin boundary are tangent, yielding the creation of islands in the basins of attraction of different attractors. Above this critical forcing amplitude the island structure becomes more pronounced, as exemplified by Fig. 4. To better visualize the role played by the critical curves in the basin boundary bifurcation, we plot both the curves and the basins of attraction. Figures 6(a)–6(d) show four cases at parameter values $a=0.001$, 0.001 42, 0.001 424, and 0.0024, respectively, where Figs. 6(a) and 6(b) are before the basin boundary bifurcation, Fig. 6(c) is very close to the basin boundary bifur-

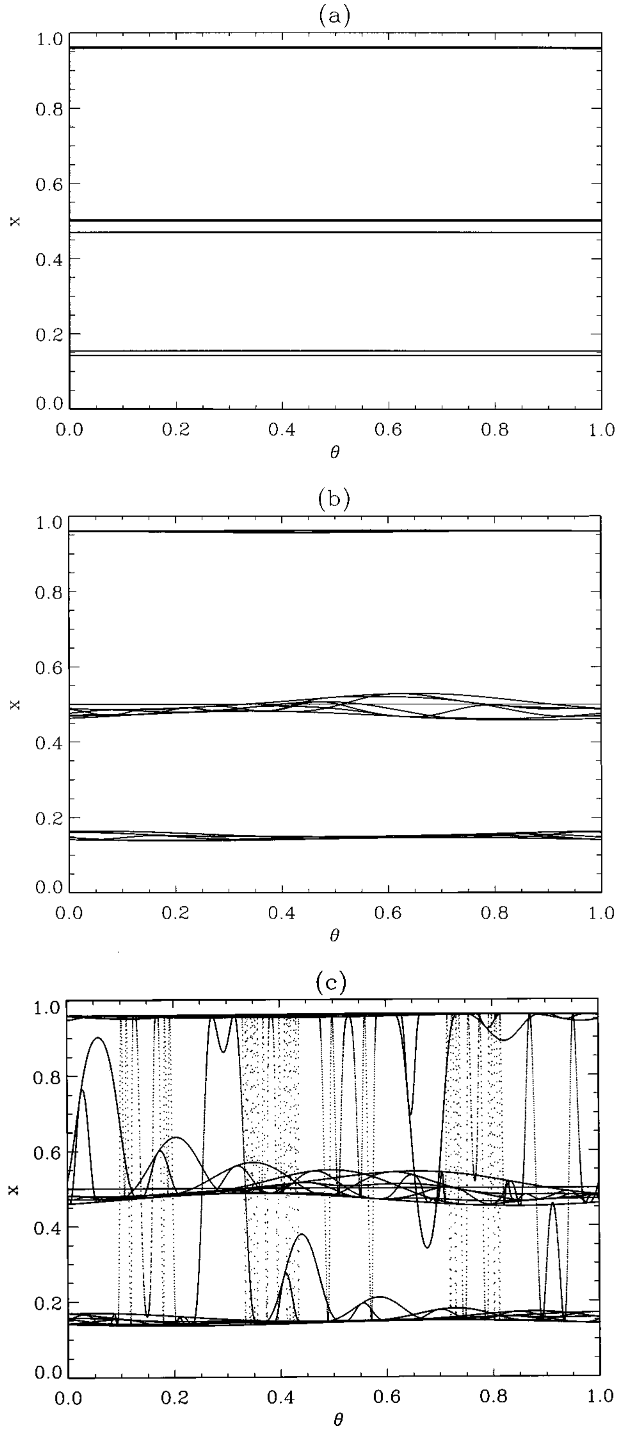

FIG. 5. Critical curves at (a) $r=3.846$ and $a=0$; (b) $r=3.846$ and $a=0.001$; and (c) $r=3.846$ and $a=0.0015$.

cation, and Fig. 6(d) is after (cf. Fig. 4).

The concept of critical curves can also be utilized to map out the parameter space regions that generate different basin structures. This is done simply by examining a large number of iterations of the critical curve $x=0.5$, $0 \leqslant \theta < 1$ to see whether they intersect the basin boundary. Figure 7 shows, in the two-dimensional parameter space (r,a), three distinct regions, where A corresponds to the parameter region before the basin boundary bifurcation where the basin structure is exemplified by Fig. 1, B corresponds to the parameter region after the basin boundary bifurcation where the basin structure is as in Fig. 4, and C denotes the parameter region in which there is only one single attractor. The boundary between regions A and B is thus the curve in the two-dimensional parameter space at which basin boundary bifurcation occurs, while the boundary between regions B and C signifies the critical parameter curve for an interior crisis [15].

As we discussed above, the basin boundary in region A is a Cantor set of invariant circles ($\mathbf{C}\times\mathbf{S}^1$) and thus their box dimension is the sum of the dimension of the Cantor set obtained from the unforced logistic map and one, the dimension of the invariant circles. Moreover, this sum does not change when the amplitude of the forcing a changes. In fact, we find that the uncertainty exponent α and, consequently, the box dimension of the boundary D is independent of a (Fig. 8). Within the accuracy of our computations, which is measured by the standard deviations of the least squares fits for each α, we obtain $\alpha=\alpha(a)=\text{const}_1$ [curve (a)] for the basin boundary which is a Cantor set of invariant circles as in Fig. 1. By contrast, in the case B, where the basins contain islands, the dimension of the basin boundary depends on the strength of forcing a. Along the line $r=3.846$ in the (r,a)-parameter space we find again $\alpha=\alpha(a)=\text{const}_2$ for those small a values for which the basin boundary is still a Cantor set of circles. Beyond the basin boundary bifurcation which occurs at $a\sim 0.001\,424$ the uncertainty exponent α appears to depend linearly on the forcing amplitude a [curve (b) in Fig. 8]. The decrease in α corresponds to an increase in the dimension of the basin boundary as we approach the interior crisis value for the attractors. This linear dependency can be understood using the same arguments as in Ref. [28] where the authors analyze a basin boundary bifurcation in a piecewise linear, noninvertible map. This map is similar to our map shown in Fig. 3 but instead of the parabolalike functions in the small rectangles they have considered piecewise linear functions. The bifurcation occurs as soon as the tips cross the boundary of the rectangle [29]. An analytical study shows that in general one obtains a power law dependence of the dimension of the basin boundary on the bifurcation parameter a: $(d-d_0)\sim(a-a_0)^\gamma$, where d_0 stands for the dimension of the boundary before the basin boundary bifurcation and a_0 denotes the forcing amplitude at the basin boundary bifurcation point. However, for the very small forcing amplitudes a applied in our example this dependency is essentially linear, which is observed in the numerical experiments.

V. DISCUSSION

Fractal and Wada basin boundaries are fundamental phenomena of deterministic chaotic systems with multiple coexisting asymptotic attractors. The basic mechanism for fractal structure to arise involves the existence of chaotic dynamics in the basin boundaries, such as the creation of nonattracting chaotic saddles in the boundaries [6]. Quasiperiodically forced systems exhibit fractal and Wada basin boundaries despite the fact that the system in neither expanding nor contracting in one direction of the motion, i.e., in the direction of the phase of the forcing. The main contribution of this paper is the detailed analysis of a unique type of basin boundary bifurcation. This basin boundary bifurcation which is related to the creation of islands in the basins of attraction

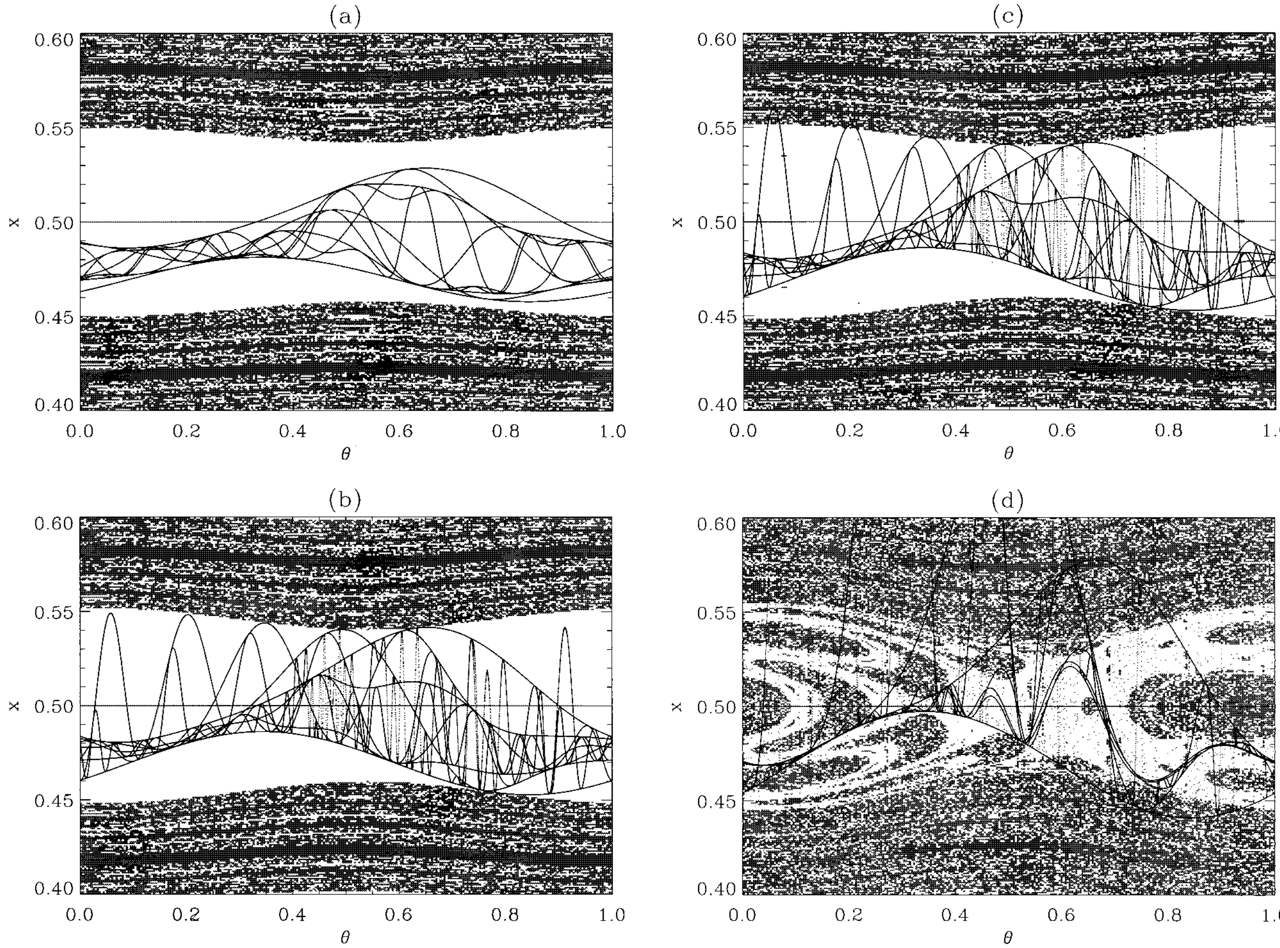

FIG. 6. Critical curves and the basin (white) of the middle attractor for $r=3.846$ and (a) $a=0.001$, (b) $a=0.001\,42$, (c) $a=0.001\,424$, and (d) $a=0.0024$. Cases (a) and (b) are before the basin boundary bifurcation, case (c) is very close to the basin boundary bifurcation, and (d) is after.

is due to the noninvertibility of the map. For this reason one cannot expect such bifurcations to happen in invertible maps and, therefore, in differential equations. However, changes in the structure of the basin boundary can also occur in quasiperiodically forced invertible maps. In such maps their appearance should be related to tangencies of stable and unstable manifolds similar to the basin boundary metamorphosis known for nonforced dissipative systems. In contrast to those, the stable and unstable manifolds involved in the basin boundary bifurcation in quasiperiodically forced systems are connected with quasiperiodic orbits (invariant curves). Furthermore, we note that the basins of attraction are symmetric with respect to the line $x=0.5$. This symme-

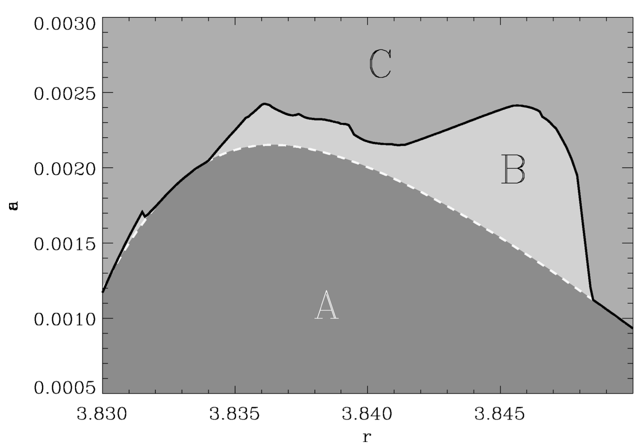

FIG. 7. Regions in the two-dimensional parameter space that generate qualitatively different basin structures.

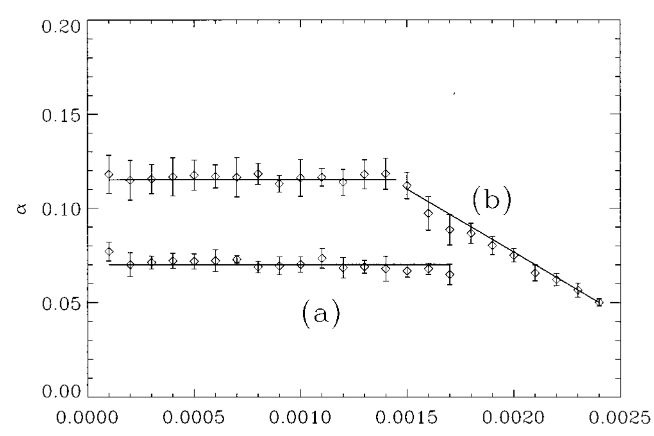

FIG. 8. The uncertainty exponent α vs the forcing amplitude a for $r=3.833$, where there is no basin boundary bifurcation up to the interior crisis value (cf. Fig. 7) (a) and $r=3.846$, where we find the maximum distance (longest scaling region) between the basin boundary bifurcation and the interior crisis value (cf. Fig. 7) (b).

try, however, is due to the symmetry of our example. In general, all results remain qualitatively the same in the absence of a specific symmetry.

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